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EFFECT OF BOUNDARY CONDITIONS ON WAVE MODELLING THROUGH FINITE DIFFERENCE METHOD

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ABSTRACT: Finding the effect of a structure with known parameters such as geometry, velocity and density underground can be defined as modelling. The purpose of the modelling is to determine the complex in-ground structure. For this, artificial seismogram models are made. Most commonly, methods based on the numerical solution of wave equations are used. Because in these methods, the source can be placed at any point of the geometry studied and instantaneous energy emanations (snap-shots) can be taken at the desired time, showing how the wave field moves in the ground. This study, it is aimed to examine the effect of boundary conditions in wave modelling with the finite difference method. For this purpose, two-dimensional scalar wave equations are modelled with the finite difference method (FDM) in the light of previously known theoretical knowledge, and how the wave field behaves in the medium using different boundary conditions is mathematically investigated. In the numerical solutions obtained with the program codes written in FORTRAN, Dirichlet and absorbing boundary conditions were used separately, and the effect of boundary conditions was revealed as snap-shot and artificial seismograms with the help of GRAPHER and MATLAB programs. From the findings, the response of the structures with smooth and simple geometry was determined with the help of seismograms and snapshots. First arrivals, reflections from the free surface, interfaces, edges and changes in energy can be observed well in the seismograms obtained as a result of the modelling made with FDM.

Keywords: Mathematical modelling; finite difference method; seismogram; boundary conditions; snap-shot

AMS Subject Classification: 35L05; 65M06; 86A15.

1. INTRODUCTION

Boundary value problems may not have exact and implicit solutions, so an approximate solution must be satisfied. Approximate methods are divided into two; some of them accept approximation in the fulfilment of boundary conditions, but it has also a necessity for being exact; in the second group of methods, although the realization of the boundary conditions is certain, there is an approximation in the provision of the differential equation. The finite difference method is a simple method that falls into this second group and can be applied to almost any situation. It is based on approximating the differential equation of the problem with the value of the function at discrete points, taking finite differences instead of differentials.

The basis of finite difference applications goes back more than two hundred years with famous scientists such as Daniel and Jacob Bernoulli, Leonard Euler, and Jacobi Stirling. Problems frequently encountered in application areas such as derivative and integration, finding inner and outer values, and fitting polynomials to numerical data can be solved with the finite difference approach [1,8-14]. In addition, wave equations can be calculated with this approach since they can be easily used in the solution of partial differential equations. In recent years, the developing computer technology and the fast and high-capacity computers have made numerical calculations more attractive and have also led to an increase in studies on finite difference approaches.

Finite difference methods can be divided into two groups: the explicit approach [2, 3] and the indirect approach [4, 5] methods. Considering numerically the wave equation applications, in the open approach; to calculate the value at a spatial point in the future, values from several points from the previous time are used, and the process is calculated for each point in succession. However, in the indirect approach, from the values of all known

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spatial points of the previous time, all points of the next time are found simultaneously with the matrix inversion method

[4]. In the analysis, it is studied with approximately 500-1000 time steps. This requires to be solved many matrices. Since this process takes a lot of time and memory, the open approach method, which can be applied more easily, has been preferred in this study.

While obtaining an artificial seismogram with the finite difference method, there are two different calculation types: homogeneous formulation and heterogeneous formulation [4,5]. In the homogeneous formulation, the elastic parameters are considered constant within each layer. In this case, boundary conditions between layers with different elastic properties must be considered [8]. In the heterogeneous formulation, these elastic properties must be specified at each grid point of the finite difference grid network and the boundary conditions must be met indirectly. Although such a formulation is very useful in modelling complex underground geometries, it requires more processes as the number of parameters increases. Therefore, in this study, a homogeneous formulation, which is easier to apply and contains fewer parameters, has been used.

Wave problems are normally solved for infinite media, but in seismogram calculations, the underground model must be limited vertically and horizontally. If appropriate boundary conditions are not used, artificial discontinuities will occur in the horizontal and vertical directions. These artificial discontinuities are called "Boundary reflections". These undesirable boundary reflections obscure the actual signals propagating in the modelled region. Therefore, infinite environments and absorbing boundary conditions are needed to prevent boundary reflections. For handling this, the wave equation is divided into right, left, and downward wave fields and the values at the boundary are determined from the plane waves going towards these boundaries. Absorbing boundary conditions for wave equations in FDM have been developed by many scientists [6]. Later, different boundary conditions were developed for Cartesian coordinates [7,8,10,12]. The advantage of Reynolds boundary conditions is that they are easy and understandable.

2. MAIN RESULTS

Solving the Two-Dimensional Wave Equation with FDM

The two-dimensional scalar wave equation is

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \tag{2.1}$$

where t is time, x and z are the distance in the horizontal and vertical directions, u is the displacement, and c is the velocity of the wave in the medium. The finite-difference representation of second-order partial derivatives is as follows.

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{(\Delta x)^2} \tag{2.2}$$

$$\frac{\partial^2 u}{\partial z^2} \approx \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{(\Delta z)^2} \tag{2.3}$$

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{(\Delta t)^2} \tag{2.4}$$

Here Δt is the time sampling interval, Δx and Δz are the sampling intervals in the x and z directions, respectively.
To make the operations easier and faster, it should be taken as $\Delta x = \Delta z$. In addition, i, j, k are indices corresponding to x (expansion direction), z (depth), and t (time) parameters, respectively.

If the finite difference equations seen in (2.2), (2.3), and (2.4) are written instead of the partial derivatives in the two-dimensional scalar wave equation (2.1), the following equation is obtained.

$$\frac{1}{c^2}\frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{(\Delta t)^2} = \frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{(\Delta x)^2} + \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{(\Delta z)^2} \tag{2.5}$$

If this equation is rearranged by taking $\lambda = \frac{c.\Delta t}{\Delta x}$ the finite difference expression of the two-dimensional wave equation is obtained.

$$u_{i,j,k+1} = 2(1-2\lambda^2)u_{i,j,k} - u_{i,j,k-1} + \lambda^2(u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k}) \quad (2.6)$$

Boundary Conditions Used in Solving the Two-Dimensional Wave Equation with FDM

In this study, two different boundary conditions are used for the environment in the two-dimensional models. These are the Dirichlet and the absorbing boundary conditions. In the solution of the scalar wave equation, the displacements are assumed to be zero at the time steps $\Delta t = 0$ and $\Delta t = 1$ as the initial condition. These conditions are suitable for the model not to be reflected at the boundaries.

$$u_{i,j,0} = u_{i,j,1} = 0 \quad (2.7)$$

Under these conditions, the two-dimensional scalar wave equation given by (2.6) will be solved for

$-a \leq x \leq a$, $0 \leq z \leq b$, and $t \geq 0$

These physical boundaries are shown in Figure 1.

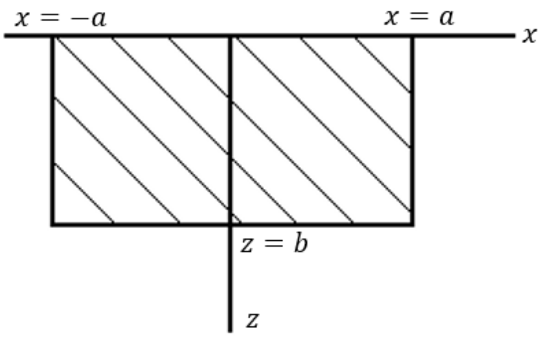


FIGURE 1 Boundaries used in solving the two-dimensional scalar wave equation

Reflection at $x = \mp a$ boundaries is not desired. In other words, the appropriate boundary conditions for the two-dimensional scalar wave not to be reflected at $x = \mp a$ and $z = b$ boundaries are as follows.

$$u(\mp a, z, t) = 0, u(x, b, t) = 0 \; or \; \frac{\partial u(\mp a,z,t)}{\partial x} = 0, \frac{\partial u(x,b,t)}{\partial x} = 0 \quad (2.8)$$

The equations seen in (2.8) are called Dirichlet boundary conditions. However, strong edge reflections are observed again at $x = \mp a$ and $z = b$ boundaries. But at these limits, the reflection coefficient is in units. For example, if the plane wave propagation to the right is considered, (2.9) is obtained.

$$u = e^{i(wt - kxcos\theta \pm kzsin\theta)}, (0 \leq \theta \leq \frac{\pi}{2}) \quad (2.9)$$

Here θ is the angle that the plane wavefront makes with the x-axis, that is, the angle of incidence of the wave to the boundary.

$$u = e^{i(wt - kxcos\theta \pm kzsin\theta)} + Re^{i(wt + kxcos\theta \pm kzsin\theta)} \quad (2.10)$$

From the displacement equation at the boundary (2.10), the reflection coefficient at $x = a$ can be calculated. If equation (2.10) is substituted in the boundary condition (2.8), the reflection coefficient $|R| = 1$ is obtained. A reflection coefficient of 1 causes the wave arriving at the boundary to be reflected with the same amplitude. These edge reflections will obscure the true reflections. To prevent these undesirable events, the wave field is divided into left, right, and downward wave fields. Therefore, the boundary condition on the left, the boundary condition on the right, the boundary condition at the base, and the boundary condition on the free surface are given by the equations seen in (2.11), (2.12), (2.13), and (2.14), respectively.

$$\left(\frac{1}{c}\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\left(\frac{\lambda}{c}\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u=0,(x=-a,0\leq z\leq b,0\leq t\leq T) \tag{2.11}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\left(\frac{\lambda}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)u=0,(x=a,0\leq z\leq b,0\leq t\leq T) \tag{2.12}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)\left(\frac{\lambda}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)u=0,(-a\leq x\leq a,0\leq z\leq b,0\leq t\leq T) \tag{2.13}$$

$$u=0,-a\leq x\leq a,z=0,0\leq t\leq T) \tag{2.14}$$

In these equations $\lambda=\frac{c.\Delta t}{\Delta x}$

If equation (2.10) is substituted in equations (2.11), (2.12), (2.13) and solved for R, the reflection coefficient becomes $R=0$. Equations (2.11), (2.12), (2.13) and (2.14) in terms of finite differences are obtained as follows.

$$u_{1,j,k+1}=u_{1,j,k}+u_{2,j,k}-u_{2,j,k-1}+\frac{c.\Delta t}{\Delta x}\left[u_{2,j,k}-u_{1,j,k}-(u_{3,j,k-1}-u_{2,j,k-1})\right], \quad 2\leq j\leq J,2\leq k\leq K \tag{2.15}$$

$$u_{I+1,j,k+1}=u_{I+1,j,k}+u_{I,j,k}-u_{I,j,k-1}+\frac{c.\Delta t}{\Delta x}\left[u_{I+1,j,k}-u_{I,j,k}-(u_{I,j,k-1}-u_{I-1,j,k-1})\right], \quad 2\leq j\leq J,2\leq k\leq K \tag{2.16}$$

$$u_{i,J+1,k+1}=u_{1,J+1,k}+u_{2,J,k}-u_{2,J,k-1}+\frac{c.\Delta t}{\Delta x}\left[u_{2,J+1,k}-u_{1,J,k}-(u_{3,J,k-1}-u_{2,J-1,k-1})\right], \quad 2\leq j\leq J,2\leq k\leq K \tag{2.17}$$

$$u_{i,1,k+1}=0,\ 2\leq i\leq I,2\leq k\leq K \tag{2.18}$$

The (2.15), (2.16), (2.17) and (2.18) equations are called absorbing boundary conditions [7]. Figure 2 shows the geometry of Dirichlet and absorbing boundary conditions in two-dimensional models.

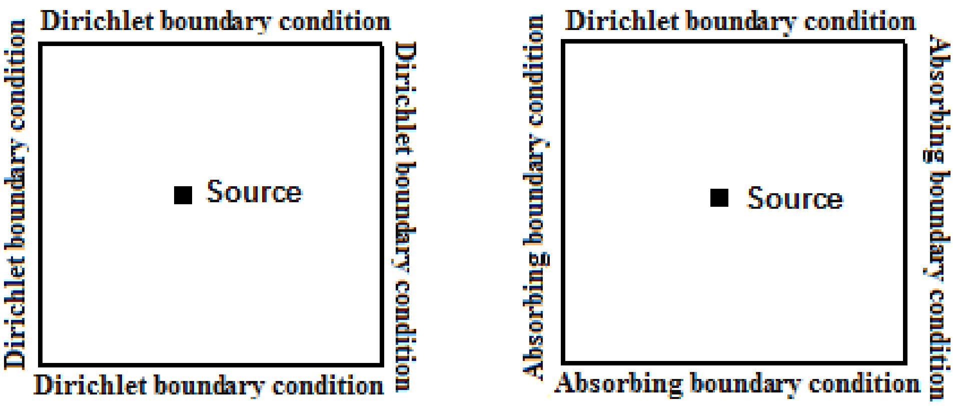


FIGURE 2 The geometry of Dirichlet and absorbing boundary conditions in two-dimensional models

3. APPLICATIONS AND ILLUSTRATIVE EXAMPLES

For a homogeneous medium with a depth of $480\,m$ and a length of $480\,m$, the velocity of the medium is $c = 1500\,m/s$ the centre frequency of the source is 30 Hz, the time sampling interval $\Delta t = 0.00236$ s. Spatial sampling intervals $\Delta x = \Delta z = h = 5\text{m}$ were taken for finite differences. The two-dimensional geological model was calculated on a 96-point grid in the horizontal and vertical directions $(96x96)$. Ricker source function is used as source function and placed in the middle of the geometry. In the models, it is assumed that the upper region is a free surface. Dirichlet boundary conditions and absorbing boundary conditions are applied as boundary conditions. Figure 3 shows the two-dimensional homogeneous underground model studied.

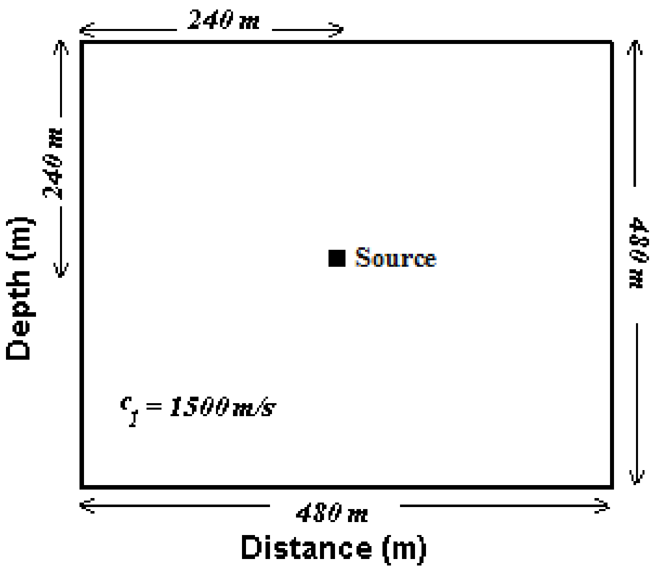


FIGURE 3 Two-dimensional homogeneous underground model

Limitations
1. In the modelling in this research, it is assumed that the environment is elastic, homogeneous and isotropic.
2. The numerical solutions of the wave equations are limited to the finite difference method.
3. The boundary conditions used are limited to Dirichlet and absorbing boundary conditions.
4. Modelling with FORTRAN programs based on finite-difference formulations of wave equations is limited to photographs and seismograms showing instantaneous energy dissipation.

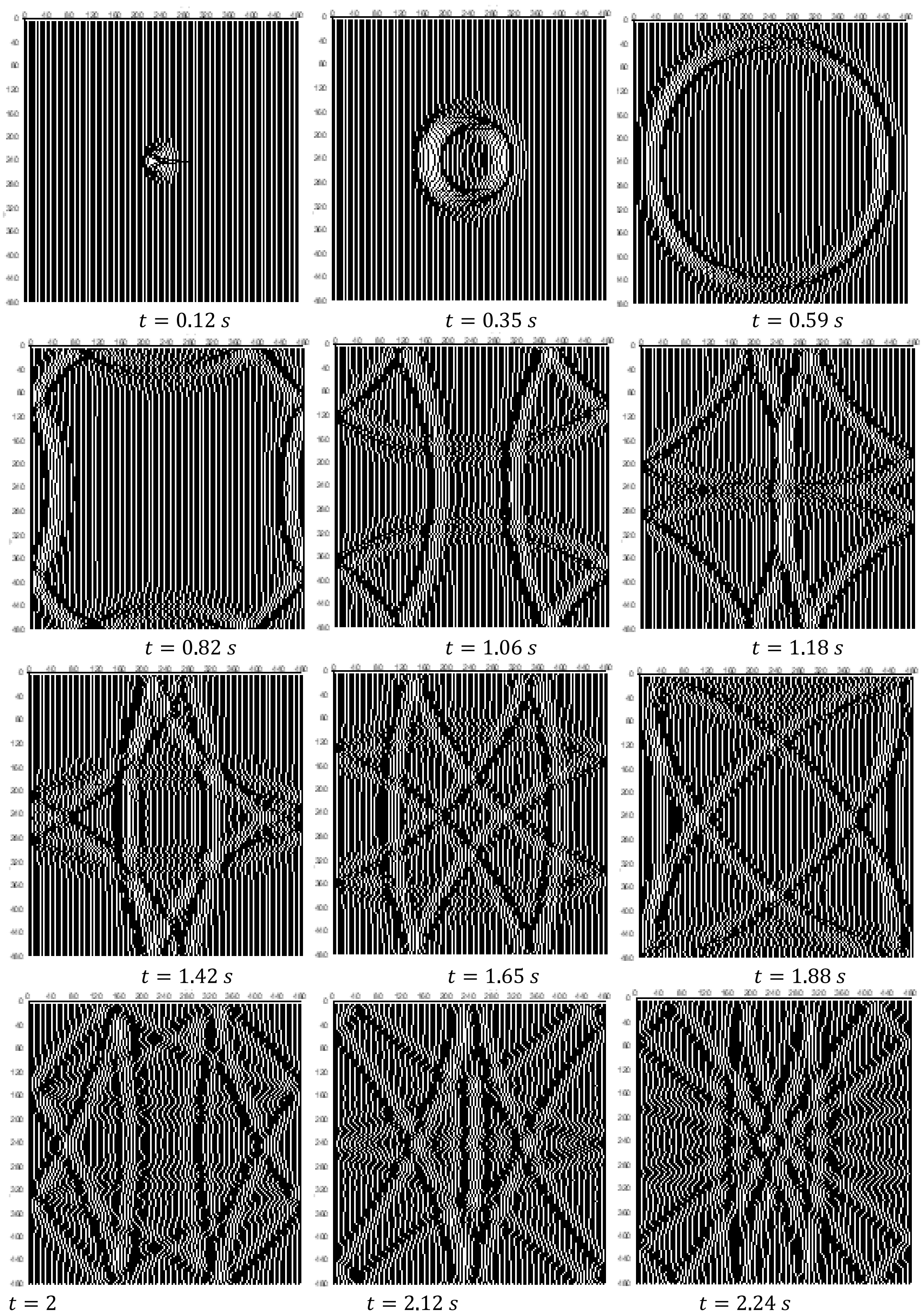


FIGURE 4 The instantaneous energy dissipations obtained using the Dirichlet boundary condition for the model is shown in Figure 3

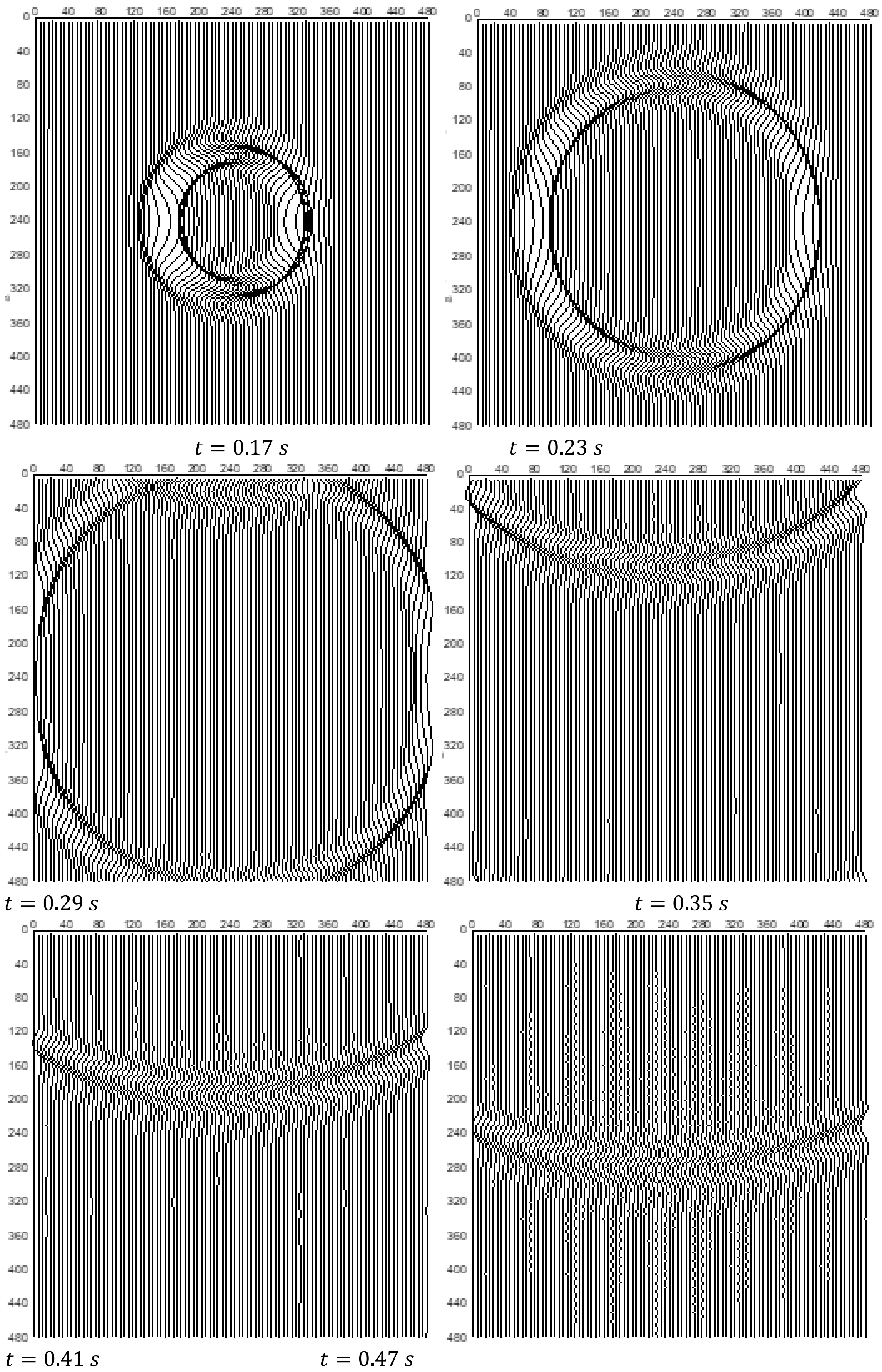
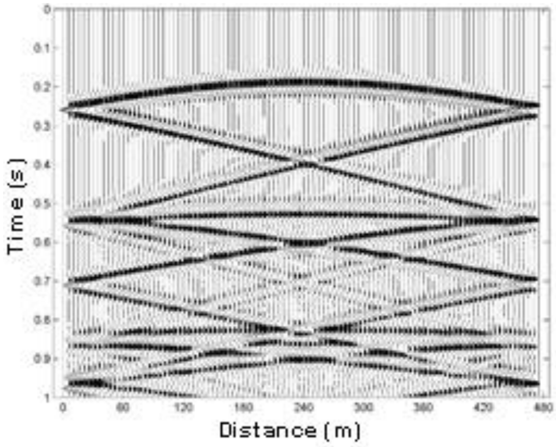
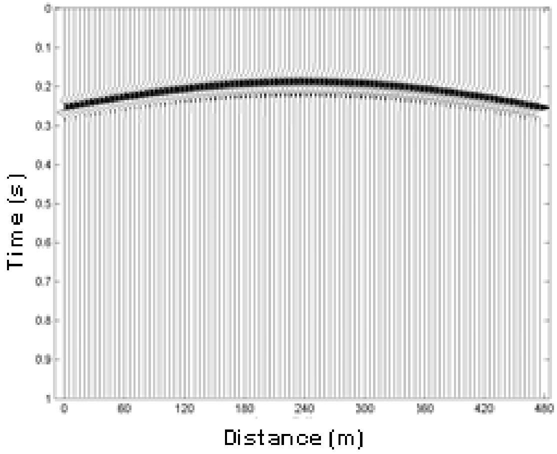


FIGURE 5 The instantaneous energy dissipations obtained using the absorbing boundary condition for the model is shown in Figure 3



a)



b)

FIGURE 6 Seismograms obtained by FDM using Dirichlet boundary conditions (a) and absorbing boundary conditions (b) for the model given in Figure 3

The wavefront propagating in a homogeneous environment can be observed from the instantaneous energy dispersions in Figure 4 and Figure 5. Since the medium is two-dimensional, the waveform propagates circularly. Naturally, in the case of a three-dimensional medium, the wavefront will be spherical. Wavefront photographs at different time steps calculated for a homogeneous medium using Dirichlet boundary conditions are shown in Figure 4. It is seen that the wavefront is reflected from the edges when it reaches the model boundaries. It is seen that the waves reflected from the edges pass through each other over time and exhibit a symmetrical appearance. This shows that the effect of the source is the same at the points that are equidistant from the source. In Figure 5, considering the absorbing boundary conditions, it is seen that there is only reflection from the free surface (earth’s surface) and no back reflection from other edges.

Figure 6. a and Figure 6. b show the seismograms obtained using Dirichlet and absorbing boundary conditions when the sensors are placed on the earth's surface. Since the source is a point $(240, 240)$, the ripple from the source is naturally recorded by the nearest receiver first. The closest receiver is the geophone, 240 m away. In addition, since the speed of the medium is 1500 m/s and the distance of the source to the nearest receiver is 240 m, the first trace is expected to reach the recorders at 240/1500=0.16 s. It is seen in the seismograms obtained in Figure 5. a and Figure 5. b that the first trace was recorded in 0.16 s by the geophone at a distance of 240 m.

4. CONCLUSION

The finite difference method is one of the most important subjects of numerical analysis, which has gained importance in recent years with its wide application area. In computer applications, a derivative or an integral cannot be calculated analytically. In this case, it is necessary to define operations in terms of calculations that the computer can do. These calculations and techniques form the scope of numerical analysis. The finite difference method consists of writing the appropriate finite difference approach instead of derivatives in differential equations.

To calculate the seismic sections of geological models in seismic studies, wave equations containing the density and velocities of the rocks are used. By comparing the obtained seismic sections with the land seismograms, it is tried to examine the underground. It is a fact that the seismic interpretation made by relying only on the records obtained from the field will be insufficient. The finite difference method helps to model by calculating artificial seismograms of complex underground models.

In this study, the two-dimensional scalar wave equation is modelled by the finite difference method. In the models made with wave equations, it is possible to see how the wave field spreads at any time and to place the source at a desired depth and distance. Depending on conditions such as stability, boundary conditions, and grid dispersion is as important as applying finite-difference approaches to wave equations. Therefore, these conditions were examined, and calculations were made by choosing the time sampling interval (Δt), distance sampling interval (Δx) and source frequency (f_p) to be used in the models under these conditions.

In a homogeneous and isotropic environment, photographs and seismograms showing instantaneous energy dissipation were obtained as a result of modelling with FORTRAN programs based on finite-difference formulations of wave equations. In these seismograms, first arrivals, reflections from the free surface, interfaces, edges, and decreases in energy are well observed. When the absorbing boundary conditions are applied, it has been observed that the reflections from the base and sides of the model disappear. It is recommended to use different approaches in wave modelling to be revealed their effects.

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